Finite Math - Fall 2016

Lecture Notes - 11/28/2016

Section 7.4 - Permutations and Combinations

There are often situations in which we have to multiply many consecutive numbers together, for example, in examples of the form "from a pool of 8 letters, make words consisting of 5 letters without any repetition." There are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ of these. Let's define a notation that will simplify writing these quantities down:

Definition 1 (Factorial)**.** *For a natural number n,*

$$\begin{aligned} n! &= n(n-1)(n-2) \cdot \cdots \cdot 2 \cdot 1 \\ 0! &= 1 \end{aligned}$$

From this definition, we can see that

$$n! = n \cdot (n-1)! = n(n-1) \cdot (n-2)! = \cdots ,$$

that is, we can explicitly write out as many of the largest numbers as we need, then write the rest as a smaller factorial. For example, we could write

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!$$

if we wanted to bring special attention to 10 through 7. The following example shows how this is useful.

Example 1. *Find*

(a) $6!$

(b) $\dfrac{10!}{9!}$

(c) $\dfrac{10!}{7!}$

(d) $\dfrac{5!}{0!3!}$

(e) $\dfrac{20!}{3!17!}$

Solution.

(a) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

(b) $\dfrac{10!}{9!} = \dfrac{10 \cdot 9!}{9!} = 10$

(c) $\dfrac{10!}{7!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = 720$

(d) $\dfrac{5!}{0!3!} = \dfrac{5 \cdot 4 \cdot 3!}{1 \cdot 3!} 5 \cdot 4 = 20$

(e) $\dfrac{20!}{3!17!} = \dfrac{20 \cdot 19 \cdot 18 \cdot 17!}{3!17!} = \dfrac{20 \cdot 19 \cdot 18}{3 \cdot 2} = 20 \cdot 19 \cdot 3 = 1140$

Example 2. *Find*

(a) $7!$

(b) $\dfrac{8!}{4!}$

(c) $\dfrac{8!}{4!(8-4)!}$

0.1. **Permutations.** Suppose we have 5 people to be seated along one side of a long table. There are many possible arrangements of the people, and each of these arrangements is called a permutation.

Definition 2 (Permutation). *A* permutation *of a set of distinct objects is an arrangement of the objects in a specific order without repetition.*

In the set up problem, we have 5 people, and 5 seats to fill. If we fill in the seats from left to right, we can put one of 5 people in the first, one of the remaining 4 in the second, one of 3 in the third, one of 2 in the fourth, and then there is only one person left to fill the fifth seat. Using the multiplication principle, we see that there are

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$$

possible arrangements, or permutations.

Theorem 1 (Permutations of n Objects). *The number of permutations of n distinct objects without repetition, denoted by ${}_nP_n$, is*

$${}_nP_n = n(n-1)\cdots 2 \cdot 1 = n!.$$

Sometimes we don't want to use all of the available options, such as when we're making 5 letter words without repetition out of a pool of 8 letters.

Definition 3 (Permutation of n Objects Taken r at a Time)**.** *A permutation of a set of n distinct objects taken r at a time without repetition is an arrangement of r of the n objects in a specific order.*

If we have n things, and we want to create a permutation using r of them we have: n choices for the first slot, $n-1$ choices for the second, $n-2$ for the third, all the way up to $n-r+1$ options for the r^{th} slot. This gives us

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

possible permutations. We can also write out the product of all numbers n though 1, then divide out the stuff we don't want: $n-r$ though 1 by using factorials

$$\begin{aligned}
\#\text{ of perm.} &= n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \\
&= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)(n-r)(n-r-1) \cdots 2 \cdot 1}{(n-r)(n-r-1) \cdots 2 \cdot 1} \\
&= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)(n-r)!}{(n-r)!} \\
&= \frac{n!}{(n-r)!}
\end{aligned}$$

Theorem 2 (Number of Permutations of n Objects Taken r at a Time)**.** *The number of permutations of n distinct objects taken r at a time without repetition is given by*

$${}_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Remark 1. *Some other notations instead of ${}_nP_r$ are P_r^n, $P_{n,r}$, and $P(n,r)$. Also note that this notation agrees with the previous formula since*

$${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!.$$

Example 3. *Given the set $\{A, B, C, D\}$, how many permutations are possible for this set of 4 objects taken 2 at a time? Answer the question using*

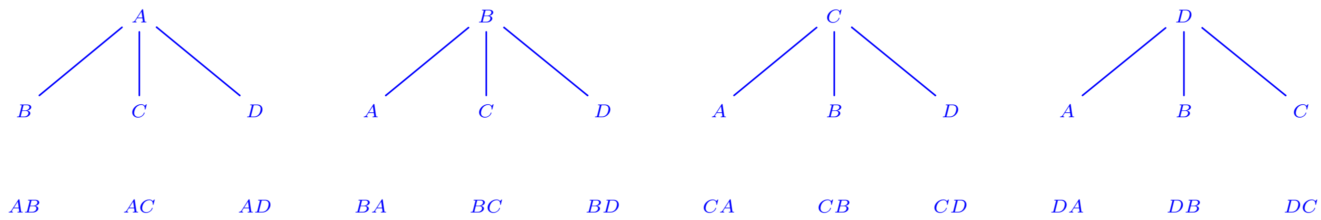
(a) *A tree diagram*

(b) *The multiplication principle*

(c) *The two formulas for ${}_nP_r$*

Solution.

(a)



So we see there are 12 *possible permutations of* 4 *objects taken* 2 *at a time.*

(b) *Using the multiplication principle, we have that there are two choices to make. The first choice has* 4 *options and the second choice has* 3 *options, thus there are* $4 \cdot 3 = 12$ *permutations.*

(c) *In this situation,* $n = 4$ *and* $r = 3$*, so* $n - r + 1 = 4 - 2 + 1 = 3$ *and*

$$_4P_2 = 4 \cdot 3 = 12.$$

Using the second formula, we have

$$_4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 4 \cdot 3 = 12.$$

Example 4. *Find the number of permutations of* 30 *objects taken* 4 *at a time.*

Solution.

$$_{30}P_4 = \frac{30!}{(30-4)!} = \frac{30!}{26!} = 30 \cdot 29 \cdot 28 \cdot 27 = 657,720$$

Example 5. *Given the set* $\{A, B, C, D\}$*, how many permutations are possible for this set of* 4 *objects taken* 3 *at a time? Answer the question using*

(a) *A tree diagram*

(b) *The multiplication principle*

(c) *The two formulas for* $_nP_r$

Example 6. *Find the number of permutations of* 67 *objects taken* 5*.*

Combinations. Suppose there is a bag that has 10 jelly beans, each with a different flavors. How many different combinations of 3 flavors can you draw from the bag? Notice that this does not take the order of the flavors into account, but the flavors themselves. In other words, if you draw cherry-grape-apple versus grape-cherry-apple, this difference is not a different combination of flavors.

Definition 4 (Combinations)**.** *A* combination *of a set of* n *distinct objects taken* r *at a time without repetition is an* r*-element subset of the set of* n *objects. The arrangement of the elements in the subset does not matter.*

If we have n objects, and we wanted to permutations of r objects at a time, we could think of that as happening in two steps:

- Step 1: Choose the r elements from among the n. This is the number of combinations that we are looking for, and we will call this number ${}_nC_r$.
- Step 2: Put the r elements into a specific order. This is just a permutation of r elements, so there are $r!$ ways to do this.

Thus, using the multiplication principle, we can see that the number of permutations of n objects taken r at a time is

$$ {}_nP_r = {}_nC_r \cdot r! $$

So, we can solve for ${}_nC_r$ to get

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}. $$

Theorem 3 (Number of Combinations of n Objects Taken r at a Time)**.** *The number of combinations of n distinct objects taken r at a time without repetition is given by*

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} $$

Remark 2. *While the textbook will stick with the notation ${}_nC_r$, a far more common notation is $\binom{n}{r}$, which is read as "n choose r." Some other notations for this are C^n_r, $C_{n,r}$, and $C(n,r)$.*

Example 7. *Form a committee of 12 people.*

(a) *In how many ways can we choose a chairperson, a vice-chairperson, a secretary, and a treasurer, assuming that one person cannot hold more than one position?*

(b) *In how many ways can we choose a subcommittee of 4 people?*

Solution.

(a) *We can this of this as choosing 4 people from among the 12 in a particular order (filling in the positions in order). This means we have*

$$ {}_{12}P_4 = \frac{12!}{(12-4)!} = \frac{12!}{8!} = 12 \cdot 11 \cdot 10 \cdot 9 = 11,880 $$

ways of filling these positions.

(b) *In this case, we just want to form a group of 4 people, so the order we choose people in is irrelevant, that means we want*

$$ {}_{12}C_4 = \frac{12!}{4!(12-4)!} = \frac{12!}{4!8!} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4!} = \frac{11,880}{24} = 495. $$

So there are 495 ways of choosing a subcommittee of 4 people from among the 12.

Example 8. *Find the number of combinations of* 30 *objects taken* 4 *at a time.*

Solution.

$$ {}_{30}C_4 = \frac{30!}{4!(30-4)!} = \frac{30!}{4!26!} = \frac{30 \cdot 29 \cdot 28 \cdot 27}{24} = 27,405 $$

Example 9. *How many ways can a* 3*-person subcommittee be selected from a committee of* 7 *people? How many ways can a president, vice-president, and secretary be chosen from a committee of* 7 *people?*

Example 10. *Find the number of combinations of* 67 *objects taken* 5 *at a time.*

Example 11. *Suppose we ave a standard* 52*-card deck and we are considering* 5*-card poker hands.*

(a) *How many hands have* 3 *hearts and* 2 *spades?*

(b) *How many hands have all the same suit? (I.e., what is the number of different flushes?)*

(c) *How many possible pairs are there? (The other three cards have a different number from the pair and each other.)*

(d) *How many possible* 3 *of a kinds are there? (The other two cards have a different number from the* 3 *of a kind and from each other.)*

(e) *How many full houses are possible? (A full house consists of a three of a kind and a pair, each from a different number.)*

Solution.

(a) *Here, there are two decisions: which* 3 *hearts to choose and which* 2 *spades to choose. Each suit has only* 13 *cards, so there are a total of*

$$ \binom{13}{3}\binom{13}{2} = 286 \cdot 78 = 22,308 $$

hands with 3 *hearts and* 2 *spades.*

(b) *This time, there are also two choices to be made. First we have to choose the suit, then we have to choose* 5 *cards from among the cards in that suit. Thus, there are*

$$ \binom{4}{1}\binom{13}{5} = 4 \cdot 1287 = 5,148 $$

possible flushes.

(c) *This time, there are several choices to make:*

(a) *Choose the number in the pair:* $\binom{13}{1}$ *choices*

(b) *Choose the suits in the pair:* $\binom{4}{2}$ *choices*

(c) *Choose the numbers of the other three cards:* $\binom{12}{3}$ *choices*

(d) *Choose the suit of the third card:* $\binom{4}{1}$ *choices*

(e) *Choose the suit of the fourth card:* $\binom{4}{1}$ *choices*

(f) *Choose the suit of the fifth card:* $\binom{4}{1}$ *choices*

This gives a total of

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 13 \cdot 6 \cdot 220 \cdot 4^3 = 1,098,240$$

possible pairs.

(d) *The number of three of a kinds is computed similarly to pairs, but is slightly easier:*

(a) *Choose the number in the triple:* $\binom{13}{1}$ *choices*

(b) *Choose the suits in the pair:* $\binom{4}{3}$ *choices*

(c) *Choose the numbers of the other two cards:* $\binom{12}{2}$ *choices*

(d) *Choose the suit of the fourth card:* $\binom{4}{1}$ *choices*

(e) *Choose the suit of the fifth card:* $\binom{4}{1}$ *choices*

This gives a total of

$$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1} = 13 \cdot 4 \cdot 66 \cdot 4^2 = 54,912$$

possible three of a kinds.

(e) *For a three of a kind, there is a pair and a three of a kind from different numbers in the hand.*

(a) *Choose the number in the three of a kind:* $\binom{13}{1}$ *choices*

(b) *Choose the suits in the three of a kind:* $\binom{4}{3}$ *choices*

(c) *Choose the number in the pair:* $\binom{12}{1}$ *choices*

(d) *Choose the suits in the pair:* $\binom{4}{2}$ *choices*

This gives a total of

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6 = 3,744$$

possible pairs.